

On the polynomial identities of the algebra $M_{11}(E)$

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Abstract

Verbally prime algebras are important in PI theory. They were described by Kemer over a field K of characteristic zero: 0 and $K\langle T \rangle$ (the trivial ones), $M_n(K)$, $M_n(E)$, $M_{ab}(E)$. Here $K\langle T \rangle$ is the free associative algebra of infinite rank, with free generators T , E denotes the infinite dimensional Grassmann algebra over K , $M_n(K)$ and $M_n(E)$ are the $n \times n$ matrices over K and over E , respectively. The algebras $M_{ab}(E)$ are subalgebras of $M_{a+b}(E)$, see their definition below. The generic (also called relatively free) algebras of these algebras have been studied extensively. Procesi described the generic algebra of $M_n(K)$ and lots of its properties. Models for the generic algebras of $M_n(E)$ and $M_{ab}(E)$ are also known but their structure remains quite unclear.

In this paper we study the generic algebra of $M_{11}(E)$ in two generators, over a field of characteristic 0 . In an earlier paper we proved that its centre is a direct sum of the field and a nilpotent ideal (of the generic algebra), and we gave a detailed description of this centre. Those results were obtained assuming the base field infinite and of characteristic different from 2 . In this paper we study the polynomial identities satisfied by this generic algebra. We exhibit a basis of its polynomial identities. It turns out that this algebra is PI equivalent to a 5-dimensional algebra of certain upper triangular matrices. The identities of the latter algebra have been studied; these were described by Gordienko. As an application of our results we describe the subvarieties of the variety of unitary algebras generated by the generic algebra in two generators of $M_{11}(E)$. Also we describe the polynomial identities in two variables of the algebra $M_{11}(E)$.

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Let K be a field of characteristic 0, and denote by E the infinite dimensional Grassmann algebra over K . If V is a K -vector space with a basis e_1, e_2, \dots then $E = E(V)$ is the vector space with a basis consisting of 1 and all products $e_{i_1}e_{i_2}\cdots e_{i_k}$, $i_1 < i_2 < \cdots < i_k$, $k \geq 1$. The multiplication in E is induced by the anticommutative law for the e_i 's, that is by $e_ie_j = -e_je_i$ for all i and j . The Grassmann algebra plays an extremely important role in the theory of PI algebras. It is one of the natural examples of a PI algebra that satisfies no standard identity whenever $\text{char}K = 0$. (Note that every PI algebra over a field of characteristic $p > 0$ satisfies some standard identity, due to a result of Kemer.) The Grassmann algebra is \mathbb{Z}_2 -graded: one can verify that $E = E_0 \oplus E_1$ where the vector subspaces E_i are spanned by all elements from its basis such that $k \equiv i \pmod{2}$. Then $E_a E_b \subseteq E_{a+b}$ where the latter sum is modulo 2. Recall that E_0 is just the centre of E . This gives E the structure of a 2-graded algebra (also called superalgebra). One of the most important developments in PI theory came with Kemer's results on the structure of the T-ideals. These results led Kemer directly to the positive solution of the long standing Specht problem, among others (see [8]). Kemer based his theory on the description of the T-prime (also called verbally prime) algebras. These turn out to be the matrix algebras $M_n(K)$, the algebras $M_n(E)$ of the matrices with entries in E , and the algebras $M_{ab}(E)$. The latter is a subalgebra of $M_{a+b}(E)$; it consists of all block matrices with blocks of sizes $a \times a$ and $b \times b$ on the main diagonal with entries from E_0 , and the remaining, off-diagonal blocks with entries from E_1 . Afterwards Kemer proved that every finitely generated PI algebra satisfies the same identities as a suitable finite dimensional algebra. Moreover, if A is any PI algebra then it satisfies the same polynomial identities as the Grassmann hull of a suitable finite dimensional superalgebra. Recall that if $A = A_0 \oplus A_1$ is 2-graded then its Grassmann hull is $G(A) = A_0 \otimes E_0 \oplus A_1 \otimes E_1$.

In spite of the importance of the T-prime algebras not much is known about the concrete form of their polynomial identities. The T-ideals of $M_n(K)$ are known only for $n = 1$ and 2 , see [16, 4], and also [17] for a streamlined version of the proof. When K is infinite and of characteristic different from 2, a basis of the identities of $M_2(K)$ is also known, see [9, 3]. The identities of the Grassmann algebra E were described in [11] in characteristic 0, and by various authors over any field, see the bibliography of [6]. In [14] Popov described the identities of $E \otimes E$ over a field of characteristic 0. Note that according to Kemer's theory $E \otimes E$ and $M_{11}(E)$ satisfy the same polynomial identities, that is they are PI equivalent. Note also that these two algebras are not PI equivalent in positive characteristic $p > 2$, see [1].

When one studies the identities of $M_n(K)$ the algebra of the generic matrices comes into play. Berele in [2] constructed generic algebras for the remaining T-prime algebras. We recall these constructions below.

In an earlier paper [10] we started studying the relatively free algebra of $M_{11}(E)$ in two generators. We were able to give quite precise description of its centre. We proved that the centre of this algebra is a direct sum of the field and a nilpotent ideal of the algebra, and we showed that the centre contains quite a lot of non-scalars. In this way we gave an answer to a question posed in [2].

Below we recall results of [10] needed for our exposition.

In this paper we work over a fixed field of characteristic 0. We describe the polynomial identities of the generic algebra in two generators for $M_{11}(E)$. It turns out the basis of its identities consists of three polynomials given in explicit form. It is worth mentioning that the same polynomials form a basis of the identities of a certain subalgebra of the 3×3 upper triangular matrices. This algebra appears in the study of extremal properties of the codimension sequence, see for example [12]. Later on its identities were described by Gordienko, see for example [7]. Our methods are based on the representation theory of the symmetric and the general linear groups, on the results from [10], and on the description of the polynomial identities of $M_{11}(E)$ given by Popov in [14]. Moreover we describe the unitary subvarieties of the variety generated by our algebra. We also prove that asymptotically the identities of every proper (unitary) subvariety behave exactly as the identities satisfied by the algebra of upper triangular matrices of order two, or as those of commutativity.

As an application of the techniques developed here we describe also the polynomial identities of $M_{11}(E)$ in two variables. To this end we use some ideas from a paper by Nikolaev, [13] where the identities in two variables of $M_2(K)$ were described.

1 Preliminaries

Throughout we consider unitary associative algebras over a field K of characteristic 0. Let $K\langle T \rangle$ be the free associative algebra freely generated over K by the set $T = \{t_1, t_2, \dots\}$. If A is a PI algebra we denote by $I = T(A) \subseteq K\langle T \rangle$ its T-ideal, that is the ideal of all identities of A . The algebra $U(A) = K\langle T \rangle / I$ is the relatively free (or generic) algebra in the variety of algebras defined by A . When T is a finite set, say $T = \{t_1, \dots, t_k\}$ one obtains the relatively free (or generic) algebra of A of rank k and denotes it by $U_k(A)$. We shall use the same letters t_i for the generators of $K\langle T \rangle$ and for their images under the canonical projection $K\langle T \rangle / T(A) = U(A)$.

We recall the construction of the free supercommutative algebra $K[X; Y]$. Let $K\langle X \cup Y \rangle$ be the free associative algebra freely generated by the set $X \cup Y$ where $X \cap Y = \emptyset$. This algebra is 2-graded in a natural way assuming the variables in X of degree 0, and those in Y of degree 1. Let I be the ideal generated by all $ab - (-1)^{|a| \cdot |b|} ba$ where a and b run over the homogeneous elements in $K\langle X \cup Y \rangle$, and $|a|$ is the \mathbb{Z}_2 -degree of the homogeneous element a , and put $K[X; Y] = K\langle X \cup Y \rangle / I$. Then $K[X; Y] \cong K[X] \otimes_K E(Y)$, here $E(Y)$ is the Grassmann algebra on the vector space with a basis Y , see for more detail [2]. If a 2-graded algebra $A = A_0 \oplus A_1$ satisfies $ab - (-1)^{|a| \cdot |b|} ba = 0$ for all homogeneous a and b then it is called supercommutative. Clearly the Grassmann algebra is supercommutative.

Take $X = \{x_{ij}^r\}$, $Y = \{y_{ij}^r\}$ where $1 \leq i, j \leq n$, $r = 1, 2, \dots$; here r is an upper index, not an exponent. One defines the matrices $A_r = (x_{ij}^r)$, $B_r = (x_{ij}^r + y_{ij}^r)$, $C_r = (z_{ij}^r)$ where $z = x$ whenever $1 \leq i, j \leq a$ or $a + 1 \leq i, j \leq a + b$,

and $z = y$ for all remaining possibilities for i and j . Suppose $a + b = n$, and consider the following subalgebras of $M_n(K[X; Y])$. The first is generated by the generic matrices A_r , $K[A_r \mid r \geq 1]$. It is isomorphic to the relatively free (or universal) algebra $U(M_n(K))$ of $M_n(K)$. In [2, Theorem 2] it was proved that $U(M_n(E)) \cong K[B_r \mid r \geq 1]$, also $U(M_{ab}(E)) \cong K[C_r \mid r \geq 1]$. The relatively free algebras of finite rank k , denoted by U_k , can be obtained by letting $r = 1, \dots, k$, that is by taking the first k matrices.

The algebra $K\langle T \rangle$ is multigraded, counting the degree of its monomials in each variable. We work over a field K of characteristic 0 therefore every T-ideal is generated by its multilinear elements, see for example [5, Section 4.2].

The polynomial identities of $M_{11}(E)$ were described by Popov in characteristic 0, see the main theorem of [14]. The theorem reads that the polynomials

$$[[t_1, t_2]^2, t_1], \quad [[t_1, t_2], [t_3, t_4], t_5] \quad (1)$$

generate the T-ideal of $E \otimes E$, and of $M_{11}(E)$ as well. Here $[a, b] = ab - ba$ is the usual commutator of a and b . The higher commutators will be left normed that is $[a, b, c] = [[a, b], c]$, and so on.

Let $L(T)$ be the free Lie algebra on the free generators T . If one substitutes the usual product in an associative algebra A by the bracket $[a, b] = ab - ba$ one gets a Lie algebra denoted by A^- . It is well known that $L(T)$ is the subalgebra of $K\langle T \rangle^-$ generated by T . Moreover $K\langle T \rangle$ is the universal enveloping algebra of $L(T)$. Choose an ordered basis of $L(T)$ consisting of T and left normed commutators. Suppose further that if u and v are elements of the basis then $\deg u < \deg v$ implies $u < v$, in this way the free generators in T precede all remaining basis elements. Then a basis of $K\langle T \rangle$ is given by 1 and all elements $t_1^{n_1} \dots t_k^{n_k} u_1 \dots u_m$ where $n_i \geq 0$, and u_i are commutators, $u_1 \leq \dots \leq u_m$. Let $B(T)$ be the subalgebra of $K\langle T \rangle$ generated by 1 and all commutators of degree at least two. Thus $B(T)$ is spanned by 1 and the products of commutators. The elements of $B(T)$ are called proper polynomials. As we work with unitary algebras it is well known that every T-ideal I is generated by its proper elements, see for example [5, Section 4.3].

2 The generic algebra of $M_{11}(E)$ in two generators

In the paper [10] we studied the generic algebra of $M_{11}(E)$ in two generators, $F = K[C_1, C_2]$ where $C_1 = \begin{pmatrix} x_1 & y_1 \\ y'_1 & x'_1 \end{pmatrix}$, $C_2 = \begin{pmatrix} x_2 & y_2 \\ y'_2 & x'_2 \end{pmatrix}$. The entries of C_1 and C_2 lie in the free supercommutative algebra $K[X; Y]$, $X = \{x_1, x_2, x'_1, x'_2\}$, $Y = \{y_1, y_2, y'_1, y'_2\}$. Here we recall some results of [10] that we need.

The algebra $K[X; Y]$ is 2-graded in a natural way. It can be given a \mathbb{Z} -grading by counting the degree in the variables of Y only: $K[X; Y] = \bigoplus_{n \in \mathbb{Z}} K[X; Y]^{(n)}$ where $K[X; Y]^{(n)}$ is the span of all monomials of degree n in the variables from

Y . Obviously $K[X; Y]^{(n)} = 0$ unless $0 \leq n \leq 4$. Also

$$\begin{aligned} K[X; Y]_0 &= K[X; Y]^{(0)} + K[X; Y]^{(2)} + K[X; Y]^{(4)}; \\ K[X; Y]_1 &= K[X; Y]^{(1)} + K[X; Y]^{(3)}. \end{aligned}$$

We set $B_0 = \{1\}$, $B_1 = \{y_1, y_2, y'_1, y'_2\}$, $B_2 = \{y_1 y_2, y_1 y'_1, y_1 y'_2, y_2 y'_1, y_2 y'_2, y'_1 y'_2\}$, $B_3 = \{y_1 y_2 y'_1, y_1 y_2 y'_2, y_1 y'_1 y'_2, y_2 y'_1 y'_2\}$, $B_4 = \{y_1 y_2 y'_1 y'_2\}$. Then for $n = 0, 1, 2, 3, 4$, $K[X; Y]^{(n)}$ is a free module over $K[X]$, with a basis B_n . As a consequence $K[X; Y]$ is a free module over $K[X]$ with a basis $B = B_0 \cup B_1 \cup B_2 \cup B_3 \cup B_4$. Also the ideals in $K[X; Y]$ are $K[X]$ -submodules.

The following elements were introduced in [10].

$$\begin{aligned} h_1 &= y_1 y_2 y'_1 y'_2; \\ h_2 &= y_1 y_2 (y'_1 (x'_2 - x_2) - y'_2 (x'_1 - x_1)); \\ h_3 &= y'_1 y'_2 (y_1 (x'_2 - x_2) - y_2 (x'_1 - x_1)); \\ h_4 &= (y'_1 (x'_2 - x_2) - y'_2 (x'_1 - x_1))(y_1 (x'_2 - x_2) - y_2 (x'_1 - x_1)). \end{aligned}$$

It is immediate to check they satisfy the following relations in $K[X; Y]$.

$$\begin{aligned} h_1 y_1 &= h_1 y'_1 = h_1 y_2 = h_1 y'_2 = 0; & h_2 y_1 &= h_2 y_2 = h_3 y'_1 = h_3 y'_2 = 0; \\ h_2 y'_1 &= h_3 y_1 = (x'_1 - x_1) h_1; & h_2 y'_2 &= h_3 y_2 = (x'_2 - x_2) h_1; \\ h_4 y_1 &= (x'_1 - x_1) h_2; & h_4 y_2 &= (x'_2 - x_2) h_2; \\ h_4 y'_1 &= -(x'_1 - x_1) h_3; & h_4 y'_2 &= -(x'_2 - x_2) h_3. \end{aligned}$$

We shall also need the polynomials

$$\begin{aligned} q_n(x_1, x'_1) &= \sum_{i=0}^n x_1^i x_1'^{n-i}; & Q_n(x_2, x'_2) &= q_n(x_2, x'_2); \\ r_n(x_1, x'_1) &= \sum_{i=0}^{n-1} (n-i) x_1^{n-1-i} x_1'^i; & R_n(x_2, x'_2) &= r_n(x_2, x'_2); \\ s_n(x_1, x'_1) &= r_n(x'_1, x_1); & S_n(x_2, x'_2) &= s_n(x_2, x'_2). \end{aligned}$$

One verifies by an obvious induction that

$$\begin{aligned} r_n &= q_{n-1} + x_1 r_{n-1}; & s_n &= q_{n-1} + x'_1 s_{n-1}; & s_n + r_n &= (n+1) q_{n-1}; \\ q_n &= x_1^n + x'_1 q_{n-1} = x_1'^n + x_1 q_{n-1}; & (x'_1 - x_1) q_{n-1} &= x_1'^n - x_1^n; \\ x_1^n x_1'^m - x_1^m x_1'^n &= (x'_1 - x_1)(q_n q_{m-1} - q_m q_{n-1}). \end{aligned}$$

We compute directly that for every m and n we have

$$\begin{aligned} C_1^m &= \begin{pmatrix} x_1^n + y_1 y'_1 r_{n-1} & y_1 q_{n-1} \\ y'_1 q_{n-1} & x_1'^n - y_1 y'_1 s_{n-1} \end{pmatrix}; \\ C_2^m &= \begin{pmatrix} x_2^m + y_2 y'_2 R_{m-1} & y_2 Q_{m-1} \\ y'_2 Q_{m-1} & x_2'^m - y_2 y'_2 S_{m-1} \end{pmatrix}. \end{aligned}$$

In this way the product $C_1^n C_2^m$ equals

$$C_1^n C_2^m = \begin{pmatrix} x_1^n x_2^m + a + d & y_1 x_2'^m q_{n-1} + y_2 x_1^n Q_{m-1} + c \\ y'_1 x_2^m q_{n-1} + y'_2 x_1'^n Q_{m-1} + c' & x_1'^n x_2'^m + a' + d' \end{pmatrix} \quad (2)$$

where $a, a' \in K[X; Y]^{(2)}$, $d, d' \in K[X; Y]^{(4)}$, and $c, c' \in K[X; Y]^{(3)}$.

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in F \cong K[C_1, C_2]$ be central. It was shown in [10] that

$$d - a \in J = \text{Ann}((x'_2 - x_2)y_1 - (x'_1 - x_1)y_2) \cap \text{Ann}((x'_2 - x_2)y'_1 - (x'_1 - x_1)y'_2).$$

By [10, Proposition 5] the $K[X]$ -module J is spanned by $\{h_1, h_2, h_3, h_4\}$. Corollary 6 from [10] states that the matrix A commutes with C_1 and C_2 if and only if $b = f_4 h_2$, $c = -f_4 h_3$, $d = a + f_1 h_1 + f_4 h_4$ for some $f_1, f_4 \in K[X]$. Hence A is central if and only if $A = aI + f_1 \begin{pmatrix} 0 & 0 \\ 0 & h_1 \end{pmatrix} + f_4 \begin{pmatrix} 0 & h_2 \\ -h_3 & h_4 \end{pmatrix}$.

Following the notation used in [10] we define matrices in F :

$$A_0 = \begin{pmatrix} h_1 & 0 \\ 0 & h_1 \end{pmatrix}; \quad A_1 = \begin{pmatrix} 0 & 0 \\ 0 & h_1 \end{pmatrix}; \quad A_2 = \begin{pmatrix} 0 & h_2 \\ -h_3 & h_4 \end{pmatrix}; \quad A_3 = \begin{pmatrix} h_4 & 0 \\ 0 & h_4 \end{pmatrix}.$$

An element $a \in F$ is *strongly central* if it is central, and also for each $b \in F$ the element ab is central in F . One checks ([10, Lemma 7]) that for arbitrary $\alpha_i \in K[X]$ the elements $\alpha_0 A_0 + \alpha_1 A_1 + \alpha_2 A_2 + \alpha_3 A_3$ are strongly central in F .

We shall need a couple of technical statements from [10]. Take a left normed commutator $f(t_1, t_2) = [t_1, t_2, t_{i_3}, \dots, t_{i_k}]$, $i_j = 1, 2$, such that $\deg_{t_1} f = n$, $\deg_{t_2} f = m$, $n + m = k$. Then Lemma 8 from [10] states that $f(C_1, C_2) = (x'_1 - x_1)^{n-1} (x'_2 - x_2)^{m-1} A(k)$ where

$$\begin{aligned} A(k) &= \begin{pmatrix} F(k) & y_1(x'_2 - x_2) - y_2(x'_1 - x_1) \\ (-1)^k (y'_2(x'_1 - x_1) - y'_1(x'_2 - x_2)) & F(k) \end{pmatrix}, \\ F(k) &= \frac{(y_1(x'_2 - x_2) - y_2(x'_1 - x_1))y'_i + (-1)^k y_i(y'_2(x'_1 - x_1) - y'_1(x'_2 - x_2))}{x'_i - x_i}. \end{aligned}$$

In the expression for $F(k)$ the index i stands for i_k . The formula for $f(C_1, C_2)$ yields that if f_1, f_2 are non-zero commutators in F then $f_1(C_1, C_2)f_2(C_1, C_2)$ is strongly central and not an identity in F , see [10, Lemma 9].

3 Some identities for the algebra F

Lemma 1 *Let $f_1(t_1, t_2)$, $f_2(t_1, t_2)$, $f_3(t_1, t_2)$ be three commutators and put $g = f_1 f_2 f_3$. Then $g(C_1, C_2)$ vanishes on F .*

Proof. We write the f_i as linear combinations of left-normed commutators, thus it suffices to consider the case when all f_i are left-normed. By [10, Lemma 9] the product $f_1(C_1, C_2)f_2(C_1, C_2)$ is, in turn, a combination of the matrices A_0, A_2, A_3 defined above. Now the entries of the matrices A_i are either zeros or, up to a sign, some of the elements h_j , $1 \leq j \leq 4$. Looking at the above expression for $f_3(C_1, C_2)$ we see that its entries vanish when multiplied by h_j . \diamond

Corollary 2 Let $f(t_1, t_2) = t_1^n t_2^m u_1^{k_1} \cdots u_r^{k_r}$ where the u_i are left-normed commutators in t_1 and t_2 . Then $f(C_1, C_2) = 0$ in F if and only if $k_1 + \cdots + k_r \geq 3$.

Proof. Lemma 1 implies the "if" part. Suppose $k_1 + \cdots + k_r \leq 2$, then the product of two left-normed commutators cannot be 0 in F . Also the matrices C_1 and C_2 are not zero divisors in F according to [10, Lemma 2]. \diamond

We make use of the following two equalities that are valid in every associative algebra. They are well known and their proofs consist of an easy induction. We separate them into a lemma for further reference.

Lemma 3 Let $z, a, b \in K\langle T \rangle$, and $n \geq 1$ then

$$[z^n, b] = \sum_{i=0}^{n-1} z^{n-i-1} [z, b] z^i; \quad [a, b] z^n = \sum_{i=0}^n \binom{n}{i} z^i [a, b, \underbrace{z, \dots, z}_{n-i}].$$

Lemma 4 Let u be a left-normed commutator in $K\langle T \rangle$ and let $v \in K\langle T \rangle$. Then $uv = \sum_i v_i u_i$ where u_i are left-normed commutators, $v_i \in K\langle T \rangle$, and $\deg u_i \geq \deg u$ for all i .

Proof. It suffices to consider $v = t_1$, then $ut_1 = [u, t_1] + t_1 u$. Then $[u, t_1]$ is left-normed and $\deg[u, t_1] > \deg u$. \diamond

Proposition 5 The polynomials $[[t_1, t_2][t_3, t_4], t_5]$ and $[t_1, t_2][t_3, t_4][t_5, t_6]$ are identities for the algebra F .

Proof. Both polynomials are multilinear therefore it is enough to evaluate them on a spanning set of the algebra F . The algebra F is spanned by elements of the type $C_1^n C_2^m u_1 \cdots u_k$ where $n, m \geq 0$ and u_i are left-normed commutators. Moreover if $k \geq 2$ then $u_1 \cdots u_k$ is strongly central hence $C_1^n C_2^m u_1 \cdots u_k$ will be central. But central elements vanish the commutators hence we consider substitutions by elements of the types $C_1^n C_2^m$ and $C_1^n C_2^m u$ only where u is a left-normed commutator.

According to Lemmas 3, 4 one has $[C_1^n C_2^m, C_1^p C_2^q] = \sum_i w_i u_i$ where the u_i are left-normed commutators, analogously for $[C_1^n C_2^m u, C_1^p C_2^q]$ and also for $[C_1^n C_2^m u, C_1^p C_2^q v]$. But then $[t_1, t_2][t_3, t_4]$ becomes $\sum_i w_i u_i v_i$ where u_i and v_i are left-normed commutators. The latter sum is (strongly) central and thus the first polynomial is an identity for F . The same procedure applied to the second polynomial yields a combination of products of three commutators which is 0 in F according to Lemma 1. \diamond

Corollary 6 The identity $[t_1, t_2, t_5][t_3, t_4] + [t_1, t_2][t_3, t_4, t_5] = 0$ holds in F .

Proof. It is another form of the first identity of Proposition 5. \diamond

Remark The polynomial $[t_1, t_2][t_3, t_4]$ is a central polynomial for $U_2(M_{11}(E))$, and in particular $[t_1, t_2]^2$ is central as well. On the other hand the latter polynomial is not central for $M_{11}(E)$. It is well known that for the matrix algebra

$M_n(K)$ a polynomial $f(t_1, \dots, t_k)$ is central if and only if $f(A_1, \dots, A_k)$ lies in the centre of the generic algebra $U_k(M_n(K))$ generated by A_1, \dots, A_k , see [15, Proposition 1.2, p. 171] This is a sharp difference in the behaviour of the generic algebras for $M_n(K)$ and for $M_{ab}(E)$.

Proposition 7 *The standard polynomial $s_4 = \sum (-1)^\sigma t_{\sigma(1)} t_{\sigma(2)} t_{\sigma(3)} t_{\sigma(4)}$ is an identity for F . Here σ runs over the permutations of the symmetric group S_4 , and $(-1)^\sigma$ stands for the sign of σ .*

Proof. First write $s_4 = [t_1, t_2] \circ [t_3, t_4] - [t_1, t_3] \circ [t_2, t_4] + [t_1, t_4] \circ [t_2, t_3]$ where $a \circ b = ab + ba$. As in Proposition 5 we shall substitute the variables by elements of the types $C_1^n C_2^m$ and $C_1^n C_2^m u$, u a left-normed commutator.

First suppose $t_1 = v_1 u$, $t_i = v_i$ where u is a left-normed commutator, and $v_i \in F$ are arbitrary. Then $[v_1 u, v_2] = v_1[u, v_2] + [v_1, v_2]u$. The product of three commutators vanishes in F hence

$$[v_1 u, v_2] \circ [v_3, v_4] = ([v_1, v_2]u) \circ [v_3, v_4] + (v_1[u, v_2]) \circ [v_3, v_4] = (v_1[u, v_2]) \circ [v_3, v_4].$$

Simple manipulations show that

$$\begin{aligned} (v_1[u, v_2]) \circ [v_3, v_4] &= v_1[u, v_2][v_3, v_4] + [v_3, v_4]v_1[u, v_2] \\ &= -v_1 u[v_3, v_4, v_2] + v_1[v_3, v_4][u, v_2] + [v_3, v_4, v_1][u, v_2] \\ &= -v_1 u[v_3, v_4, v_2] - v_1[v_3, v_4, v_2]u - [v_3, v_4, v_1, v_2]u. \end{aligned}$$

Then one obtains by the identity of Jacobi

$$\begin{aligned} s_4(v_1 u, v_2, v_3, v_4) &= -v_1 u([v_3, v_4, v_2] - [v_2, v_4, v_3] + [v_2, v_3, v_4]) \\ &\quad - v_1([v_3, v_4, v_2] - [v_2, v_4, v_3] + [v_2, v_3, v_4])u \\ &\quad - ([v_3, v_4, v_1, v_2] - [v_2, v_4, v_1, v_3] + [v_2, v_3, v_1, v_4])u \\ &= -([v_3, v_4, v_1, v_2] - [v_2, v_4, v_1, v_3] + [v_2, v_3, v_1, v_4])u. \end{aligned}$$

The latter sum equals, once again by Jacobi,

$$([v_1, v_2], [v_4, v_3]) + ([v_1, v_3], [v_2, v_4]) + ([v_1, v_4], [v_3, v_2])u$$

and this vanishes as a combination of products of three commutators.

Now we consider the substitution of t_i by $C_1^{m_i} C_2^{m_i}$, $1 \leq i \leq 4$. First one defines on $K[X; Y]$ an automorphism $'$ of order two by $x_i \mapsto x'_i$, $y_i \mapsto y'_i$, and then extending to the whole supercommutative algebra. It is easy to see that for every $D = (d_{ij}) \in F$ it holds $d_{22} = d'_{11}$ and $d_{21} = d'_{12}$. This notation agrees also with the formula for the product $C_1^n C_2^m$ in (2). Suppose $D_i = C_1^{m_i} C_2^{m_i}$. Hence we may consider $D_i = \begin{pmatrix} a_i & b_i \\ b'_i & a'_i \end{pmatrix}$, for some $a_i \in K[X; Y]_0$ and $b_i \in K[X; Y]_1$.

Put $D = (d_{ij}) = s_4(D_1, D_2, D_3, D_4) \in F$; according to the above it suffices to prove that $d_{11} = d_{21} = 0$.

Direct computation shows that the $(1, 1)$ -entry of $[D_1, D_2] \circ [D_3, D_4]$ is

$$\begin{aligned} & 2(b_1b'_2 + b'_1b_2)(b_3b'_4 + b'_3b_4) \\ & + (b_1(a'_2 - a_2) - b_2(a'_1 - a_1))(b'_4(a'_3 - a_3) - b'_3(a'_4 - a_4)) \\ & + (b_3(a'_4 - a_4) - b_4(a'_3 - a_3))(b'_2(a'_1 - a_1) - b'_1(a'_2 - a_2)). \end{aligned}$$

Now writing down the analogous expressions for the remaining two summands of s_4 as above, and summing up we have

$$\begin{aligned} d_{11} = & 2(b_1b'_2 + b'_1b_2)(b_3b'_4 + b'_3b_4) - 2(b_1b'_3 + b'_1b_3)(b_2b'_4 + b'_2b_4) \\ & + 2(b_1b'_4 + b'_1b_4)(b_2b'_3 + b'_2b_3) \\ & + (b_1(a'_2 - a_2) - b_2(a'_1 - a_1))(b'_4(a'_3 - a_3) - b'_3(a'_4 - a_4)) \\ & + (b_3(a'_4 - a_4) - b_4(a'_3 - a_3))(b'_2(a'_1 - a_1) - b'_1(a'_2 - a_2)) \\ & - (b_1(a'_3 - a_3) - b_3(a'_1 - a_1))(b'_4(a'_2 - a_2) - b'_2(a'_4 - a_4)) \\ & - (b_2(a'_4 - a_4) - b_4(a'_2 - a_2))(b'_3(a'_1 - a_1) - b'_1(a'_3 - a_3)) \\ & + (b_1(a'_4 - a_4) - b_4(a'_1 - a_1))(b'_3(a'_2 - a_2) - b'_2(a'_3 - a_3)) \\ & + (b_2(a'_3 - a_3) - b_3(a'_2 - a_2))(b'_4(a'_1 - a_1) - b'_1(a'_4 - a_4)). \end{aligned}$$

The last six expressions above cancel altogether, and we are left with

$$d_{11} = 4(b'_1b'_2b_3b_4 - b'_1b'_3b_2b_4 + b'_1b'_4b_2b_3 + b'_2b'_3b_1b_4 - b'_2b'_4b_1b_3 + b'_3b'_4b_1b_2).$$

As $D_i = C_1^{n_i}C_2^{m_i}$ it follows by (2) that $b_i = x_1^{n_i}Q_{m_i-1}y_2 + x_2^{m_i}q_{n_i-1}y_1 + c_i$ for some $c_i \in K[X; Y]^{(3)}$. Also $b'_ib'_jb'_kb'_l = g(i, j, k, l)y_1y_2y'_1y'_2$ where

$$\begin{aligned} g(i, j, k, l) = & x_1^{n_l}x_1^{m_j}q_{n_k-1}q_{n_i-1}x_2^{m_i}x_2^{m_k}Q_{m_j-1}Q_{m_l-1} \\ & + x_1^{n_k}x_1^{m_i}q_{n_j-1}q_{n_l-1}x_2^{m_j}x_2^{m_l}Q_{m_i-1}Q_{m_k-1} \\ & - x_1^{n_k}x_1^{m_j}q_{n_l-1}q_{n_i-1}x_2^{m_i}x_2^{m_l}Q_{m_j-1}Q_{m_k-1} \\ & - x_1^{n_l}x_1^{m_i}q_{n_j-1}q_{n_k-1}x_2^{m_j}x_2^{m_k}Q_{m_i-1}Q_{m_l-1}. \end{aligned}$$

Thus for $d_{11} = s_4(D_1, D_2, D_3, D_4)_{11}$ we have

$$\begin{aligned} d_{11} = & (g(1, 2, 3, 4) - g(1, 3, 2, 4) + g(1, 4, 2, 3))y_1y_2y'_1y'_2 \\ & + (g(2, 3, 1, 4) - g(2, 4, 1, 3) + g(3, 4, 1, 2))y_1y_2y'_1y'_2. \end{aligned}$$

Expanding the sum of the $g(i, j, k, l)$ above we arrive at

$$\begin{aligned} & Q_{m_2-1}Q_{m_3-1}q_{n_1-1}q_{n_4-1}(x_2^{m_4}x_2^{m_1} - x_2^{m_1}x_2^{m_4})(x_1^{n_3}x_1^{n_2} - x_1^{n_2}x_1^{n_3}) \\ & + Q_{m_1-1}Q_{m_4-1}q_{n_2-1}q_{n_3-1}(x_2^{m_3}x_2^{m_2} - x_2^{m_2}x_2^{m_3})(x_1^{n_4}x_1^{n_1} - x_1^{n_1}x_1^{n_4}) \\ & + Q_{m_2-1}Q_{m_4-1}q_{n_1-1}q_{n_3-1}(x_2^{m_1}x_2^{m_3} - x_2^{m_3}x_2^{m_1})(x_1^{n_4}x_1^{n_2} - x_1^{n_2}x_1^{n_4}) \\ & + Q_{m_1-1}Q_{m_3-1}q_{n_2-1}q_{n_4-1}(x_2^{m_2}x_2^{m_4} - x_2^{m_4}x_2^{m_2})(x_1^{n_3}x_1^{n_1} - x_1^{n_1}x_1^{n_3}) \\ & + Q_{m_1-1}Q_{m_2-1}q_{n_3-1}q_{n_4-1}(x_2^{m_4}x_2^{m_3} - x_2^{m_3}x_2^{m_4})(x_1^{n_2}x_1^{n_1} - x_1^{n_1}x_1^{n_2}) \\ & + Q_{m_3-1}Q_{m_4-1}q_{n_1-1}q_{n_2-1}(x_2^{m_1}x_2^{m_2} - x_2^{m_2}x_2^{m_1})(x_1^{n_3}x_1^{n_4} - x_1^{n_4}x_1^{n_3}). \end{aligned}$$

By the relations from Section 2: $(x_1^n x_1'^m - x_1^m x_1'^n) = (x_1' - x_1)(q_n q_{m-1} - q_m q_{n-1})$ and $(x_2^n x_2'^m - x_2^m x_2'^n) = (x_2' - x_2)(Q_n Q_{m-1} - Q_m Q_{n-1})$ it follows $d_{11} = 0$.

Now we prove that $s_4(D_1, D_2, D_3, D_4)_{21} = d_{21} = 0$. The approach is similar to that of d_{11} . Computing the $(2, 1)$ -entry of $[D_1, D_2] \circ [D_3, D_4]$ we obtain

$$\begin{aligned} & 2(b_2'(a_1' - a_1) - b_1'(a_2' - a_2))(b_3 b_4' + b_3' b_4) \\ & + 2(b_4'(a_3' - a_3) - b_3'(a_4' - a_4))(b_1 b_2' + b_1' b_2) \\ & = 2(a_1' - a_1)(b_2' b_3 b_4 + b_2 b_3' b_4') - 2(a_2' - a_2)(b_1' b_3' b_4 + b_1 b_3 b_4') \\ & + 2(a_3' - a_3)(b_1 b_2' b_4' + b_1' b_2 b_4) - 2(a_4' - a_4)(b_1' b_2 b_3' + b_1 b_2' b_3). \end{aligned}$$

Permuting the indices we get that $(1/4)d_{21}$ equals

$$\begin{aligned} & (a_1' - a_1)(b_2' b_3 b_4 - b_2' b_4 b_3 + b_3' b_4' b_2) - (a_2' - a_2)(b_1' b_3' b_4 - b_1' b_4' b_3 + b_3' b_4' b_1) \\ & + (a_3' - a_3)(b_1' b_2' b_4 - b_1' b_4' b_2 + b_2' b_4' b_1) - (a_4' - a_4)(b_1' b_2' b_3 - b_1' b_3' b_2 + b_2' b_3' b_1). \end{aligned}$$

Substitute $b_i = x_1^{n_i} Q_{m_i-1} y_2 + x_2^{m_i} q_{n_i-1} y_i + c_i$, and $a_i = x_1^{n_i} x_2^{m_i} + d_i + e_i$ as in the previous case. Here $c_i \in K[X; Y]^{(3)}$, $d_i \in K[X; Y]^{(2)}$, and $e_i \in K[X; Y]^{(4)}$.

One has $b_i' b_j' b_k = f_0(i, j, k) y_1 y_1' y_2' + g_0(i, j, k) y_2 y_1' y_2'$ with

$$\begin{aligned} f_0(i, j, k) &= x_2^{m_k} q_{n_k-1} (x_2^{m_i} Q_{m_j-1} x_1^{n_j} q_{n_i-1} - x_2^{m_j} Q_{m_i-1} x_1^{n_i} q_{n_j-1}) \\ g_0(i, j, k) &= x_1^{n_k} Q_{m_k-1} (x_2^{m_i} Q_{m_j-1} x_1^{n_j} q_{n_i-1} - x_2^{m_j} Q_{m_i-1} x_1^{n_i} q_{n_j-1}). \end{aligned}$$

By applying the equality $(x_2^n x_2'^m - x_2^m x_2'^n) = (x_2' - x_2)(Q_n Q_{m-1} - Q_m Q_{n-1})$ and after manipulation we get that $f_0(i, j, k) - f_0(i, k, j) + f_0(j, k, i)$ equals

$$\begin{aligned} & q_{n_k-1} q_{n_i-1} Q_{m_j-1} x_1^{n_j} (x_2^{m_k} x_2^{m_i} - x_2^{m_i} x_2^{m_k}) \\ & + q_{n_k-1} q_{n_j-1} Q_{m_i-1} x_1^{n_i} (x_2^{m_j} x_2^{m_k} - x_2^{m_k} x_2^{m_j}) \\ & + q_{n_j-1} q_{n_i-1} Q_{m_k-1} x_1^{n_k} (x_2^{m_i} x_2^{m_j} - x_2^{m_j} x_2^{m_i}) \\ & = Q_{m_i} Q_{m_j-1} Q_{m_k-1} q_{n_i-1} (q_{n_k-1} x_1^{n_j} - q_{n_j-1} x_1^{n_k}) (x_2' - x_2) \\ & + Q_{m_i-1} Q_{m_j} Q_{m_k-1} q_{n_j-1} (q_{n_i-1} x_1^{n_k} - q_{n_k-1} x_1^{n_i}) (x_2' - x_2) \\ & + Q_{m_i-1} Q_{m_j-1} Q_{m_k} q_{n_k-1} (q_{n_j-1} x_1^{n_i} - q_{n_i-1} x_1^{n_j}) (x_2' - x_2). \end{aligned}$$

Thus $d_{21} = 4f(x) y_1 y_1' y_2' + 4g(x) y_2 y_1' y_2'$. We shall prove $f(x) = g(x) = 0$. But

$$\begin{aligned} (1/4)f(x) &= (x_1^{n_1} x_2^{m_1} - x_1^{n_1} x_2^{m_1})(f_0(2, 3, 4) - f_0(2, 4, 3) + f_0(3, 4, 2)) \\ &- (x_1^{n_2} x_2^{m_2} - x_1^{n_2} x_2^{m_2})(f_0(1, 3, 4) - f_0(1, 4, 3) + f_0(3, 4, 1)) \\ &+ (x_1^{n_3} x_2^{m_3} - x_1^{n_3} x_2^{m_3})(f_0(1, 2, 4) - f_0(1, 4, 2) + f_0(2, 4, 1)) \\ &- (x_1^{n_4} x_2^{m_4} - x_1^{n_4} x_2^{m_4})(f_0(1, 2, 3) - f_0(1, 3, 2) + f_0(2, 3, 1)). \end{aligned}$$

But $(x_1^n x_2^m - x_1^m x_2^n) = x_1^n (x_2' - x_2) Q_{m-1} + x_2^m (x_1' - x_1) q_{n-1}$; substituting the f_0 by their defining equalities in $f_0(i, j, k) - f_0(i, k, j) + f_0(j, k, i)$ we get

$f(x) = (x'_2 - x_2)^2 f^{(1)}(x) + (x'_2 - x_2)(x'_1 - x_1) f^{(2)}(x)$. Here

$$\begin{aligned}
f^{(1)}(x) = & Q_{m_1-1} Q_{m_2} Q_{m_3-1} Q_{m_4-1} x_1^{l_{n_1}} q_{n_2-1} (q_{n_4-1} x_1^{l_{n_3}} - q_{n_3-1} x_1^{l_{n_4}}) \\
& + Q_{m_1-1} Q_{m_2-1} Q_{m_3} Q_{m_4-1} x_1^{l_{n_1}} q_{n_3-1} (q_{n_2-1} x_1^{l_{n_4}} - q_{n_4-1} x_1^{l_{n_2}}) \\
& + Q_{m_1-1} Q_{m_2-1} Q_{m_3-1} Q_{m_4} x_1^{l_{n_1}} q_{n_4-1} (q_{n_3-1} x_1^{l_{n_2}} - q_{n_2-1} x_1^{l_{n_3}}) \\
& - Q_{m_1} Q_{m_2-1} Q_{m_3-1} Q_{m_4-1} x_1^{l_{n_2}} q_{n_1-1} (q_{n_4-1} x_1^{l_{n_3}} - q_{n_3-1} x_1^{l_{n_4}}) \\
& - Q_{m_1-1} Q_{m_2-1} Q_{m_3} Q_{m_4-1} x_1^{l_{n_2}} q_{n_3-1} (q_{n_1-1} x_1^{l_{n_4}} - q_{n_4-1} x_1^{l_{n_1}}) \\
& - Q_{m_1-1} Q_{m_2-1} Q_{m_3-1} Q_{m_4} x_1^{l_{n_2}} q_{n_4-1} (q_{n_3-1} x_1^{l_{n_1}} - q_{n_1-1} x_1^{l_{n_3}}) \\
& + Q_{m_1} Q_{m_2-1} Q_{m_3-1} Q_{m_4-1} x_1^{l_{n_3}} q_{n_1-1} (q_{n_4-1} x_1^{l_{n_2}} - q_{n_2-1} x_1^{l_{n_4}}) \\
& + Q_{m_1-1} Q_{m_2} Q_{m_3-1} Q_{m_4-1} x_1^{l_{n_3}} q_{n_2-1} (q_{n_1-1} x_1^{l_{n_4}} - q_{n_4-1} x_1^{l_{n_1}}) \\
& + Q_{m_1-1} Q_{m_2-1} Q_{m_3-1} Q_{m_4} x_1^{l_{n_3}} q_{n_4-1} (q_{n_2-1} x_1^{l_{n_1}} - q_{n_1-1} x_1^{l_{n_2}}) \\
& - Q_{m_1} Q_{m_2-1} Q_{m_3-1} Q_{m_4-1} x_1^{l_{n_4}} q_{n_1-1} (q_{n_3-1} x_1^{l_{n_2}} - q_{n_2-1} x_1^{l_{n_3}}) \\
& - Q_{m_1-1} Q_{m_2} Q_{m_3-1} Q_{m_4-1} x_1^{l_{n_4}} q_{n_2-1} (q_{n_1-1} x_1^{l_{n_3}} - q_{n_3-1} x_1^{l_{n_1}}) \\
& - Q_{m_1-1} Q_{m_2-1} Q_{m_3-1} Q_{m_4} x_1^{l_{n_4}} q_{n_3-1} (q_{n_2-1} x_1^{l_{n_1}} - q_{n_1-1} x_1^{l_{n_2}}).
\end{aligned}$$

After simple manipulations we obtain $f^{(1)}(x) = 0$. As for $f^{(2)}(x)$ we have

$$\begin{aligned}
f^{(2)}(x) &= q_{n_1-1}x_2^{m_1}Q_{m_2}Q_{m_3-1}Q_{m_4-1}q_{n_2-1}(q_{n_4-1}x_1^{n_3} - q_{n_3-1}x_1^{n_4}) \\
&+ q_{n_1-1}x_2^{m_1}Q_{m_2-1}Q_{m_3}Q_{m_4-1}q_{n_3-1}(q_{n_2-1}x_1^{n_4} - q_{n_4-1}x_1^{n_2}) \\
&+ q_{n_1-1}x_2^{m_1}Q_{m_2-1}Q_{m_3-1}Q_{m_4}q_{n_4-1}(q_{n_3-1}x_1^{n_2} - q_{n_2-1}x_1^{n_3}) \\
&- q_{n_2-1}x_2^{m_2}Q_{m_1}Q_{m_3-1}Q_{m_4-1}q_{n_1-1}(q_{n_4-1}x_1^{n_3} - q_{n_3-1}x_1^{n_4}) \\
&- q_{n_2-1}x_2^{m_2}Q_{m_1-1}Q_{m_3}Q_{m_4-1}q_{n_3-1}(q_{n_1-1}x_1^{n_4} - q_{n_4-1}x_1^{n_1}) \\
&- q_{n_2-1}x_2^{m_2}Q_{m_1-1}Q_{m_3-1}Q_{m_4}q_{n_4-1}(q_{n_3-1}x_1^{n_1} - q_{n_1-1}x_1^{n_3}) \\
&+ q_{n_3-1}x_2^{m_3}Q_{m_1}Q_{m_2-1}Q_{m_4-1}q_{n_1-1}(q_{n_4-1}x_1^{n_2} - q_{n_2-1}x_1^{n_4}) \\
&+ q_{n_3-1}x_2^{m_3}Q_{m_1-1}Q_{m_2}Q_{m_4-1}q_{n_2-1}(q_{n_1-1}x_1^{n_4} - q_{n_4-1}x_1^{n_1}) \\
&+ q_{n_3-1}x_2^{m_3}Q_{m_1-1}Q_{m_2-1}Q_{m_4}q_{n_4-1}(q_{n_2-1}x_1^{n_1} - q_{n_1-1}x_1^{n_2}) \\
&- q_{n_4-1}x_2^{m_4}Q_{m_1}Q_{m_2-1}Q_{m_3-1}q_{n_1-1}(q_{n_3-1}x_1^{n_2} - q_{n_2-1}x_1^{n_3}) \\
&- q_{n_4-1}x_2^{m_4}Q_{m_1-1}Q_{m_2}Q_{m_4-1}q_{n_2-1}(q_{n_1-1}x_1^{n_3} - q_{n_3-1}x_1^{n_1}) \\
&- q_{n_4-1}x_2^{m_4}Q_{m_1-1}Q_{m_2-1}Q_{m_4}q_{n_3-1}(q_{n_2-1}x_1^{n_1} - q_{n_1-1}x_1^{n_2}) \\
&= q_{n_1-1}q_{n_2-1}q_{n_3-1}Q_{m_4-1}x_1^{n_4}x_2^{m_1}(Q_{m_2-1}Q_{m_3} - Q_{m_2}Q_{m_3-1}) \\
&+ q_{n_1-1}q_{n_2-1}q_{n_3-1}Q_{m_4-1}x_1^{n_4}x_2^{m_2}(Q_{m_1}Q_{m_3-1} - Q_{m_1-1}Q_{m_3}) \\
&+ q_{n_1-1}q_{n_2-1}q_{n_3-1}Q_{m_4-1}x_1^{n_4}x_2^{m_3}(Q_{m_1-1}Q_{m_2} - Q_{m_1}Q_{m_2-1}) \\
&- q_{n_1-1}q_{n_2-1}q_{n_4-1}Q_{m_3-1}x_1^{n_3}x_2^{m_1}(Q_{m_2-1}Q_{m_4} - Q_{m_2}Q_{m_4-1}) \\
&- q_{n_1-1}q_{n_2-1}q_{n_4-1}Q_{m_3-1}x_1^{n_3}x_2^{m_2}(Q_{m_1}Q_{m_4-1} - Q_{m_1-1}Q_{m_4}) \\
&- q_{n_1-1}q_{n_2-1}q_{n_4-1}Q_{m_3-1}x_1^{n_3}x_2^{m_4}(Q_{m_1-1}Q_{m_2} - Q_{m_1}Q_{m_2-1}) \\
&+ q_{n_1-1}q_{n_3-1}q_{n_4-1}Q_{m_2-1}x_1^{n_2}x_2^{m_1}(Q_{m_3-1}Q_{m_4} - Q_{m_3}Q_{m_4-1}) \\
&+ q_{n_1-1}q_{n_3-1}q_{n_4-1}Q_{m_2-1}x_1^{n_2}x_2^{m_3}(Q_{m_1}Q_{m_4-1} - Q_{m_1-1}Q_{m_4}) \\
&+ q_{n_1-1}q_{n_3-1}q_{n_4-1}Q_{m_2-1}x_1^{n_2}x_2^{m_4}(Q_{m_1-1}Q_{m_3} - Q_{m_1}Q_{m_3-1}) \\
&- q_{n_2-1}q_{n_3-1}q_{n_4-1}Q_{m_1-1}x_1^{n_1}x_2^{m_2}(Q_{m_3-1}Q_{m_4} - Q_{m_3}Q_{m_4-1}) \\
&- q_{n_2-1}q_{n_3-1}q_{n_4-1}Q_{m_1-1}x_1^{n_1}x_2^{m_3}(Q_{m_2}Q_{m_4-1} - Q_{m_2-1}Q_{m_4}) \\
&- q_{n_2-1}q_{n_3-1}q_{n_4-1}Q_{m_1-1}x_1^{n_1}x_2^{m_4}(Q_{m_2-1}Q_{m_3} - Q_{m_2}Q_{m_3-1}).
\end{aligned}$$

Applying $(x_2^n x_2'^m - x_2'^n x_2^m) = (x_2' - x_2)(Q_n Q_{m-1} - Q_m Q_{n-1})$ one sees that the first three summands in the final expression cancel out. Repeat the procedure for the remaining three groups of 3 summands in each thus getting $f^{(2)}(x) = 0$.

The computation needed to show $g(x) = 0$ is quite similar to the above (also in length), and we omit it. Thus s_4 is an identity for F . \diamond

Remark Neither of the identities from Propositions (5), (7) is a polynomial identity for $M_{11}(E)$. Clearly $T(M_{11}(E)) \subseteq T(F)$; it can be easily shown that all identities for $M_{11}(E)$ follow from $[[t_1, t_2][t_3, t_4], t_5]$ (though the latter is not an identity for $M_{11}(E)$).

4 The identities of $M_{11}(E)$

Here we shall prove that the polynomials from Propositions (5), (7) generate the T-ideal of F . To this end we make use of the results of Popov [14]. Namely we shall need not only the concrete form of the basis of $T(M_{11}(E))$ but also the structure of the corresponding relatively free algebra. In order to make our exposition more self-contained we recall here the results from [14].

Let Γ_n be the vector space of the proper multilinear polynomials in t_1, \dots, t_n in the free associative algebra $K\langle T \rangle$. We work with unitary algebras over a field of characteristic 0 hence the T-ideal of any algebra A is generated by the intersections $T(A) \cap \Gamma_n$. The vector space Γ_n is a left module over the symmetric group S_n . The action of S_n on Γ_n is by permuting the variables. We refer the reader to [5, Chapter 12] for all necessary information concerning the representations of S_n and their applications to PI theory.

Let W be the variety of algebras defined by $M_{11}(E)$ and put $\Gamma_n(W) = \Gamma_n / (\Gamma_n \cap T(W))$. Then clearly $\Gamma_n(W)$ is an S_n -module (with the induced action). The result from [14] we need is the decomposition of $\Gamma_n(W)$ into irreducible submodules. It is well known that the irreducibles for S_n are described in terms of partitions of n and Young tableaux. We refer once again to [5] for that description. Recall that the polynomial representations of the general linear group GL_m are also described in terms of partitions (of not more than m parts) and Young diagrams, see [5]. Sometimes it is convenient to use the "symmetrized" version of a generator of an irreducible S_n -module; it generates an irreducible GL_m -module, and vice versa via linearisation. Recall that one can linearise and go back as $\text{char} K = 0$.

The following polynomials were defined in [14]. Let $p \geq q \geq 2$ and $s \geq 0$. Set $\varphi_p^{(s)} = \varphi_p^{(s)}(t_1, \dots, t_p)$ and $\varphi_{p,q}^{(s)} = \varphi_{p,q}^{(s)}(t_1, \dots, t_p)$ as follows.

$$\begin{aligned} \varphi_p^{(s)} &= \begin{cases} \sum_{\sigma \in S_p} (-1)^\sigma [t_{\sigma(1)}, t_{\sigma(2)}] \cdots [t_{\sigma(p)}, x_1^{(s)}] & p \text{ odd,} \\ \sum_{\sigma \in S_p} (-1)^\sigma [t_{\sigma(1)}, t_{\sigma(2)}] \cdots [t_{\sigma(p-1)}, t_{\sigma(p)}, t_1^{(s)}] & p \text{ even,} \end{cases} \\ \varphi_{p,q}^{(s)} &= \begin{cases} \sum_{\tau \in S_q} (-1)^\tau [t_{\tau(1)}, t_{\tau(2)}] \cdots [t_{\tau(q-1)}, t_{\tau(q)}] \varphi_p^{(s)} & q \text{ even,} \\ \sum_{\sigma \in S_p} (-1)^\sigma [t_{\sigma(1)}, t_{\sigma(2)}] \cdots [t_{\sigma(p-1)}, t_{\sigma(p)}] \varphi_q^{(s)} & p \text{ even, } q \text{ odd} \end{cases} \end{aligned}$$

and $\varphi_{p,q}^{(s)} = \sum (-1)^{\sigma\tau} [x_{\tau(1)}, x_{\tau(2)}] \cdots [x_{\sigma(1)}, x_{\sigma(2)}] \cdots [x_{\sigma(p)}, x_1^{(s)}, x_{\tau(q)}]$ when both p and q are odd. Here the summation runs over $\sigma \in S_p$, $\tau \in S_q$, and $[a, b^{(s)}]$ stands for $[a, b, \dots, b]$ with s entries of b .

Set $M_p^{(s)}$ to be the S_n -submodule of $\Gamma_n(W)$ generated by $\varphi_p^{(s)}$, $n = p + s$, and $M_{p,q}^{(s)}$ the S_n -submodule of $\Gamma_n(W)$ generated by $\varphi_{p,q}$, $n = p + q + s$.

Let $f(t_1, \dots, t_n) \in \Gamma_n$ be a proper multilinear polynomial and suppose $d = D_\alpha$ is a Young tableau associated to a diagram D of a partition of n . That is we fill the boxes of the diagram D , along the rows, with the numbers of the permutation α of n . Form the Young semi-idempotent $e(d)$ of d and denote by $M(d, f)$ the S_n -module generated by $e(d)f$. It is well known that it is either 0 or irreducible.

The polynomials $\varphi_p^{(s)}$ and $\varphi_{p,q}^{(s)}$ are not multilinear but we consider their complete linearisations. The description of $\Gamma_n(W)$ given in [14] is based on the following results.

1. If $p \geq 2$, $s \geq 0$ then $\varphi_p^{(s)}$ and $\varphi_{p,q}^{(s)}$ are not polynomial identities for $M_{11}(E)$.
2. Let $n = p + s$ and let $D = (s+1, 1^{p-1})$ be a partition of n with an associated Young tableau $d = D_\alpha$. If $f(t_1, \dots, t_n) \in \Gamma_n(W)$ then $M(d, f) \subseteq M_p^{(s)}$.
3. Let $n = p + q + s$, $d = D_\alpha$ with $D = (s+2, 2^{q-1}, 1^{p-q})$ and $f = f(x_1, \dots, x_n) \in \Gamma_n(W)$. Then $M(d, f) \subseteq M_{p,q}^{(s)}$.
4. If the second row of a diagram D contains at least 3 boxes then for every f the module $M(d, f)$ is 0 in $\Gamma_n(W)$.
5. The decomposition of $\Gamma_n(W)$ in irreducibles is

$$\Gamma_n(W) = (\oplus_{p+s=n} M_p^{(s)}) \bigoplus (\oplus_{p+q+s=n} M_{p,q}^{(s)}).$$

Popov deduced, as a corollary to the above listed results, the main theorem of [14], namely that the T-ideal of $M_{11}(E)$ is generated by the polynomials $[[t_1, t_2]^2, t_1]$ and $[t_1, t_2, [t_3, t_4], t_5]$.

5 The identities of F

Theorem 8 *Let K be a field of characteristic 0. The polynomials*

$$[[t_1, t_2][t_3, t_4], t_5], \quad [t_1, t_2][t_3, t_4][t_5, t_6], \quad s_4(t_1, t_2, t_3, t_4) \quad (3)$$

form a basis of the polynomial identities for the algebra $F = K[C_1, C_2]$.

Proof. First we shall prove that the polynomials $\varphi_4^{(s)}$ lie in the T-ideal generated by the three polynomials from our theorem. Recall that

$$\varphi_p^{(s)} = \begin{cases} \sum_{\sigma \in S_p} (-1)^\sigma [t_{\sigma(1)}, t_{\sigma(2)}] \cdots [t_{\sigma(p)}, x_1^{(s)}] & p \text{ odd}, \\ \sum_{\sigma \in S_p} (-1)^\sigma [t_{\sigma(1)}, t_{\sigma(2)}] \cdots [t_{\sigma(p-1)}, t_{\sigma(p)}, t_1^{(s)}] & p \text{ even}. \end{cases}$$

Also it is immediate that $\varphi_4^{(0)}(t_1, t_2, t_3, t_4) = 4s_4(t_1, t_2, t_3, t_4)$. Suppose $s \geq 1$. By the identity from Corollary 6 we obtain that $(1/4)\varphi_p^{(s)}(t_1, t_2, t_3, t_4)$ equals

$$\begin{aligned} & [t_1, t_2][t_3, t_4, t_1^{(s)}] + (-1)^s [t_3, t_4, t_1^{(s)}][t_1, t_2] + [t_2, t_3][t_1, t_4, t_1^{(s)}] \\ & + (-1)^s [t_1, t_4, t_1^{(s)}][t_2, t_3] - [t_2, t_4][t_1, t_3, t_1^{(s)}] - (-1)^s [t_1, t_3, t_1^{(s)}][t_2, t_4]. \end{aligned}$$

Thus $(1/4)\varphi_p^{(s)}(t_1, t_1 t_2, t_3, t_4)$ will be equal to

$$\begin{aligned} & [t_1, t_3]t_2[t_1, t_4, t_1^{(s)}] + [t_1, t_4][t_1, t_3, t_1^{(s)}]t_2 - [t_1, t_4]t_2[t_1, t_3, t_1^{(s)}] \\ & - [t_1, t_3][t_1, t_4, t_1^{(s)}]t_2 - [t_1, t_4][t_2, t_3, t_1^{(s+1)}] - [x_3, x_4][x_1, x_2, x_1^{(s+1)}] \\ & + [t_1, t_3][t_2, t_4, t_1^{(s+1)}] + (1/4)t_1\varphi_4^{(s)}(t_1, t_2, t_3, t_4). \end{aligned}$$

Analogously for $(1/4)\varphi_p^{(s)}(t_1, t_2t_1, t_3, t_4)$ we obtain

$$\begin{aligned} & t_2[t_1, t_3][t_1, t_4, t_1^{(s)}] + (-1)^s[t_1, t_4, t_1^{(s)}]t_2[t_1, t_3] - t_2[t_1, t_4][t_1, t_3, t_1^{(s)}] \\ & - (-1)^s[t_1, t_3, t_1^{(s)}]t_2[t_1, t_4] - [t_1, t_2][t_3, t_4, t_1^{(s+1)}] - [t_2, t_3][t_1, t_4, t_1^{(s+1)}] \\ & + [t_2, t_4][t_1, t_3, t_1^{(s+1)}] + (1/4)\varphi_4^{(s)}(t_1, t_2, t_3, t_4)t_1. \end{aligned}$$

Therefore for $(1/4)(\varphi_p^{(s)}(t_1, t_1t_2, t_3, t_4) + \varphi_p^{(s)}(t_1, t_2t_1, t_3, t_4))$ we have

$$\begin{aligned} & (1/4)(t_1 \circ \varphi_4^{(s)}(t_1, t_2, t_3, t_4) - \varphi_4^{(s+1)} + [t_1, t_3]t_2[t_1, t_4, t_1^{(s)}] \\ & - (-1)^s[t_1, t_3, t_1^{(s)}]t_2[t_1, t_4] - [t_1, t_4]t_2[t_1, t_3, t_1^{(s)}] + (-1)^s[t_1, t_4, t_1^{(s)}]t_2[t_1, t_3]). \end{aligned}$$

An easy computation shows that the first two identities from the theorem, together with the fact that commutation by a fixed element is a derivation, imply the identity $[[t_1, t_2]t_3[t_4, t_5], t_6] = 0$. Using the fact that the product of 3 commutators is 0 we obtain $[t_1, t_2]t_3[t_4, t_5, t_6] + [t_1, t_2, t_6]t_3[t_4, t_5] = 0$. Repeating several times we arrive at

$$\begin{aligned} & [[t_1, t_3]t_2[t_1, t_4, t_1^{(s)}] - (-1)^s[t_1, t_3, t_1^{(s)}]t_2[t_1, t_4]] = 0, \\ & [t_1, t_4]t_2[t_1, t_3, t_1^{(s)}] - (-1)^s[t_1, t_4, t_1^{(s)}]t_2[t_1, t_3] = 0. \end{aligned}$$

In this way $\varphi_4^{(s+1)}(t_1, t_2, t_3, t_4)$ equals

$$\varphi_4^{(s)}(t_1, t_2, t_3, t_4) \circ t_1 - \varphi_4^{(s)}(t_1, t_1t_2, t_3, t_4) - \varphi_4^{(s)}(t_1, t_2t_1, t_3, t_4),$$

and we may proceed by induction on s since $\varphi_4^{(0)} = 4s_4$.

As we work with unitary algebras and $\text{char}K = 0$ we can consider only the multilinear proper identities. Denote by I the T-ideal generated by the identities from the theorem and let V be the variety of unitary algebras determined by I ; we shall study the S_n -module $\Gamma_n(V) = \Gamma_n/\Gamma_n \cap I$. But the T-ideal of $M_{11}(E)$ is contained in I hence $\Gamma_n(V)$ is a homomorphic image of $\Gamma_n(W)$, and we have to determine which of the irreducibles in $\Gamma_n(W)$ vanish modulo the T-ideal I .

But it is easy to see that the polynomials $\varphi_p^{(s)}$, $p \geq 5$, are products of at least three commutators, and as such they follow from the second identity of the theorem. The same holds for $\varphi_{p,q}^{(s)}$ whenever $p + q \geq 5$. Also we showed above that $\varphi_4^{(s)}$ lies in I for every $s \geq 0$. Therefore

$$\Gamma_n(V) = M_2^{(n-2)} \oplus M_3^{(n-3)} \oplus M_{2,2}^{(n-4)}.$$

Hence in order to complete the proof it suffices to see that the generators of the irreducible modules from the above decomposition are all non-zero modulo I . But $\varphi_2^{(s)}(t_1, t_2) = 2[t_1, t_2, t_1^{(s)}]$ and $\varphi_2^{(s)}(C_1, C_2) \neq 0$ according to [10, Lemma 9].

Analogously $\varphi_3^{(s)}(t_1, t_2, t_3) = 2([t_1, t_2][t_3, t_1^{(s)}] - [t_1, t_3][t_2, t_1^{(s)}])$ and we obtain $\varphi_3^{(s)}(C_1, C_2, [C_1, C_2]) = -4[C_1, C_2][C_2, C_1^{(s+1)}] \neq 0$ due to [10, Lemma 9].

Finally $\varphi_{2,2}^{(s)} = 4[t_1, t_2][t_1, t_2, t_1^{(s)}]$ and $\varphi_{2,2}^{(s)}(C_1, C_2) = 4[C_1, C_2][C_1, C_2, C_1^{(s)}]$ is non-zero for the same reason as above. \diamond

In [7], A. Gordienko studied the identities satisfied by the algebra $A_1 \subseteq UT_3(K)$ where $UT_3(K)$ are the upper triangular matrices of order 3 over K . The algebra A_1 consists of all matrices whose $(1, 1)$ and $(3, 3)$ entries are equal. He deduced that $T(A_1)$ is generated by the same three identities as in our Theorem 8. Moreover Gordienko described basis of the vector space $P_n(A_1)$ and $\Gamma_n(A_1)$ of the multilinear and the proper multilinear elements of degree n modulo the identities of A_1 , respectively. Combining our result with Gordienko's theorem we obtain the following corollary.

Corollary 9 *The algebras $F = K[C_1, C_2]$ and A_1 are PI equivalent over a field of characteristic 0.*

It will be interesting to know whether these two algebras remain PI equivalent if the field K is infinite and of characteristic $p > 2$.

6 A description of the subvarieties

In this section we shall describe the subvarieties of the variety V of unitary algebras defined by the identities of the algebra F . We shall work over a field of characteristic 0. Recall that $\Gamma_n(V) = M_2^{(n-2)} \oplus M_3^{(n-3)} \oplus M_{2,2}^{(n-4)}$. Therefore we have to find the consequences of degree $n+1$ in $\Gamma_{n+1}(V)$ of the polynomials $\varphi_2^{(n-2)}$, $\varphi_{2,2}^{(n-4)}$, $\varphi_3^{(n-3)}$.

Proposition 10 *The consequences of degree $n+1$ are as follows.*

- (1) *The polynomials $\varphi_2^{(n-1)}$, $\varphi_{2,2}^{(n-3)}$, and $\varphi_3^{(n-2)}$, follow from $\varphi_2^{(n-2)}$.*
- (2) *The polynomials $\varphi_{2,2}^{(n-3)}$ and $\varphi_3^{(n-2)}$ follow from $\varphi_{2,2}^{(n-4)}$.*
- (3) *The polynomials $\varphi_{2,2}^{(n-3)}$ and $\varphi_3^{(n-2)}$ follow from $\varphi_3^{(n-3)}$.*

Proof. Let $u_2^{(n-2)}(t_1, t_2, t_3)$ be the multihomogeneous component of the polynomial $\varphi_2^{(n-2)}(t_1 + t_3, t_2)$ that is linear in t_3 , and let $u_{2,2}^{(n-4)}(t_1, t_2, t_3)$ be the component that is linear in t_2 and in t_3 of $\varphi_{2,2}^{(n-4)}(t_1, t_2 + t_3)$. Clearly the identities $u_2^{(n-2)}$ and $\varphi_2^{(n-2)}$ are equivalent, and also $u_{2,2}^{(n-4)}$ and $\varphi_{2,2}^{(n-4)}$ are equivalent, as the former are partial linearisations of the latter. Then

$$\begin{aligned} u_2^{(n-2)} &= -2([t_2, t_3, t_1^{(n-2)}] + (n-3)[t_2, t_1, t_3, t_1^{(n-3)}] + [t_2, t_1^{(n-2)}, t_3]); \\ u_{2,2}^{(n-4)} &= -4([t_1, t_2][t_3, t_1^{(n-3)}] + [t_1, t_3][t_2, t_1^{(n-3)}]). \end{aligned}$$

In order to prove the proposition we verify with the generators of the irreducible modules of $\Gamma_{n+1}(M_{1,1})$ are consequences of each generator of the irreducible modules of $\Gamma_n(M_{1,1})$.

1. We compute directly that $\varphi_2^{(n-1)} = \varphi_2^{(n-2)}t_1 - t_1\varphi_2^{(n-2)}$. Similarly

$$\begin{aligned}\varphi_3^{(n-2)}(t_1, t_2, t_3) &= (-1)^n(u_2^{(n-2)}(t_1, t_3, t_1t_2) - u_2^{(n-2)}(t_1, t_2, t_1t_3) \\ &\quad + t_1(u_2^{(n-2)}(t_1, t_2, t_3) - u_2^{(n-2)}(t_1, t_3, t_2)))/(n-2) \\ &\quad + (n-1)(\varphi_2^{(n-2)}(t_1, t_2)t_3 - \varphi_2^{(n-2)}(t_1, t_3)t_2)).\end{aligned}$$

Also $\varphi_{2,2}^{(n-3)} = u_2^{(n-2)}(t_1, t_1t_2, t_2) + 2\varphi_2^{(n-2)}t_2 - t_1u_2^{(n-2)}(t_1, t_2, t_2)$ when n is odd, and similarly $\varphi_{2,2}^{(n-3)} = (u_2^{(n-1)}(t_1, t_2, t_2) - [\varphi_2^{(n-2)}(t_1, t_2), t_2])/(n-2) - u_2^{(n-2)}(t_1, t_2, [t_1, t_2])$ if n is even.

2. One computes directly that

$$\begin{aligned}\varphi_{2,2}^{(n-3)} &= t_1u_{2,2}^{(n-4)}(t_1, t_2, t_2) - u_{2,2}^{(n-4)}(t_1, t_1t_2, t_2); \\ \varphi_3^{(n-3)} &= (u_{2,2}^{(n-4)}(t_1, t_2, t_1t_3) - u_{2,2}^{(n-4)}(t_1, t_1t_2, t_3))/2.\end{aligned}$$

Moreover $\varphi_2^{(n-1)}$ is not a consequence of $\varphi_{2,2}^{(n-4)}$. In order to see this observe that $\varphi_{2,2}^{(n-4)}(C_1, C_2)$ is strongly central in F . Hence all its consequences will be central but $\varphi_2^{(n-2)}(C_1, C_2)$ is not central.

3. Finally, for $\varphi_{2,2}^{n-4}$, one checks directly that

$$\begin{aligned}\varphi_3^{(n-2)} &= t_1\varphi_3^{(n-3)} + \varphi_3^{(n-3)}t_1 - \varphi_3^{(n-3)}(t_1, t_1t_2, t_3) - \varphi_3^{(n-3)}(t_1, t_2t_1, t_3); \\ \varphi_{2,2}^{(n-3)} &= \varphi_3^{(n-3)}(t_1, t_2, [t_1, t_2]).\end{aligned}$$

As in (2), $\varphi_2^{(n-1)}$ is not a consequence of $\varphi_3^{(n-3)}$. ◇

Let I and J be two T-ideals in $K\langle X \rangle$. Then I and J are asymptotically equal if for all sufficiently large n it holds $I \cap P_n = J \cap P_n$. As we consider unitary algebras we can substitute P_n by Γ_n .

Corollary 11 *If U is a proper subvariety of V then U is asymptotically equivalent to either $\text{var}(K)$ or to $\text{var}(UT_2(K))$.*

Proof. The variety U satisfies, for some n , at least one of the identities $\varphi_2^{(n-2)}(x_1, x_2)$, $\varphi_3^{(n-3)}(x_1, x_2, x_3)$, $\varphi_{2,2}^{(n-4)}(x_2, x-2)$.

If it satisfies some $\varphi_2^{(n-2)}$ then by Proposition 10 every commutator of sufficiently large degree will vanish on U . Therefore U is asymptotically equivalent to $\text{var}(K)$. (Recall the latter is generated by the identity $[t_1, t_2]$.)

If on the other hand U satisfies $\varphi_{2,2}^{(n-4)}$ or $\varphi_3^{(n-3)}$ then every proper polynomial of sufficiently large degree, that is a product of two commutators, will vanish on U . But the T-ideal of $UT_2(K)$ is generated by the product of two commutators, hence V is asymptotically equivalent to $\text{var}(UT_2(K))$. ◇

7 Identities in two variables for $M_{11}(E)$

In this section we apply some of the results obtained above in order to describe the identities in two variables for the algebra $M_{11}(E)$ over a field of characteristic 0. Recall that in this way we determine the identities in two variables for $E \otimes E$ as well. The identities in two variables for $M_2(K)$, $\text{char} K = 0$ were described by Nikolaev in [13]. We shall employ some of the results obtained in [13].

A word about the notation we shall use here. In order not to accumulate indices we shall use the letters x and y for the variables.

The main theorem in [13] states that the identities in two variables for $M_2(K)$ all follow from the Hall polynomial $h(x, y) = [[x, y]^2, x]$. Recall that $T(M_2(K))$ is generated by h and by s_4 ; one usually writes $h(x, y, z) = [[x, y]^2, z]$. This form of h is equivalent to the one above *modulo* the standard polynomial s_4 , see for example [3]; otherwise the two forms of the Hall polynomial are not equivalent.

Let H be the variety of (unitary) algebras determined by the polynomial h . As in the previous section we shall work with the proper multilinear elements only. Define

$$f_{kmn} = [x, y]^k [x, y, x^{(m)}, y^{(n)}]; \quad d_{k,l}(x, y) = [x, y]^k [x, y, x^{(l)}].$$

Nikolaev proved that the polynomials f_{kmn} , $k, m, n \geq 0$ span the vector space of all proper multilinear polynomials modulo the identity h . Moreover he proved that $\Gamma_n(H) = \oplus M_{k,l}$ where the sum is over all $k, l \geq 0$ such that $2k + l + 2 = n$. The irreducible S_n -modules $M_{k,l}$ are generated by the complete linearisations of the polynomials $d_{k,l}$ given above.

But $h(x, y) = [[x, y]^2, x]$ vanishes on $M_{1,1}(E)$. Therefore $\Gamma_n(M_{11}(E))$ is a homomorphic image of $\Gamma_n(H)$. We denote by J the T-ideal of the identities in two variables for $M_{11}(E)$, thus we will have a decomposition $\Gamma_n(M_{1,1}) = \oplus M_{k,l} \pmod{J}$. Here the sum is taken over (some of the) k and l . Thus we have to check which of the polynomials $d_{k,l}$ are identities for $M_{11}(E)$ and which are not.

Lemma 12 *The polynomial $d_{k,l}$ is not an identity for $M_{1,1}(E)$ when $k < 2$.*

Proof. Suppose $k = 0$, then $d_{0,l} = [x, y, x^{(l)}]$, and $d_{0,l}(C_1, C_2) \neq 0$. Therefore $d_{0,l}$ is not an identity for $M_{11}(E)$. If $k = 1$ we have $d_{1,l} = [x, y][x, y, x^{(l)}]$ and as above $d_{1,l}$ is not an identity for $M_{11}(E)$. \diamond

Lemma 13 *The polynomials $d_{k,l}$, $k \geq 2$, are identities for $M_{1,1}(E)$. If $k \geq 3$ then $d_{k,l}$ follows from $d_{2,l}$.*

Proof. Suppose $k \geq 2$, then for every $l \geq 0$ the polynomial $d_{k,l}$ is a product of three commutators. Therefore $d_{k,l}(C_1, C_2) = 0$ and consequently $d_{k,l} \in T(M_{11}(E))$. The second statement is immediate. \diamond

Thus we have the following corollary.

Corollary 14 *All identities in two variables for $M_{11}(E)$ are consequences from $[[x, y]^2, x]$ and the polynomials $d_{2,l} = [x, y]^2 [x, y, x^{(l)}]$, $l \geq 0$.*

Theorem 15 *All identities in two variables for $M_{11}(E)$ follow from the two polynomials $h = [[x, y]^2, x]$ and $d = [x, y]^3$.*

Proof. Clearly h and d are identities for $M_{11}(E)$. In order to prove the theorem it suffices to show, according to Lemma 13, that all polynomials $d_{2,l} = [x, y]^2[x, y, x^{(l)}]$, are consequences of d and h . We shall induct on l . The base of the induction is $l = 0$ when $d_{2,0} = d$. Write then

$$\begin{aligned} [x, y]^2[x, y, x^{(l)}] &= [x, y]^2([x, y, x^{(l-1)}]x - x[x, y, x^{(l-1)}]) \\ &= [x, y]^2[x, y, x^{(l-1)}]x - [x, y]^2x[x, y, x^{(l-1)}]. \end{aligned}$$

As $[x, y]^2$ commutes with x (and with y) we have that $d_{2,l}$ follows from $d_{2,l-1}$ and we are done. \diamond

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